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The conditions for the uniform validity of three time scale approximation on a very long time interval

by

Lih-Chyun Wang

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CHAPTER 1. INTRODUCTION

The method of multiple scales is one of the most popular ways of creating approximate solutions of differential equations containing a small perturbation parameter. An approximate solution derived in a straightforward manner, such as by formally expanding in power series in the perturbation parameter often fails in that the resulting approximation may not satisfy all conditions of the problem such as boundary and initial conditions, boundedness, etc. Such problems are of singular perturbation type and new procedures are called for. Poincaré in his investigations of the motions of the planets, recognized this problem and used new, scaled, variables to eliminate certain unbounded terms which arose in his approximate solution of initial value problems. Such methods have been refined in recent years, extended to other situations and now come under the heading of the method of multiple scales.

A somewhat different approach to the approximate solution of certain initial value problems is the method of averaging. Unlike the method of multiple scales, the method of averaging has more theoretic results to support the method. The approximate solutions created by the method of averaging, in general, we can expect to be uniformly valid on an interval of order $O(\frac{1}{\epsilon})$ [1; Theorem 6.3.1. p. 294].

The aim of the 3-scale method is more ambitious. It is not only to improve the asymptotic order of the error estimate, but also to extend the validity of the approximation to a longer interval of time, that is, an expanding interval of length $O(\frac{1}{\epsilon^2})$ or longer. In other words, the 3-scale method is an attempt to capture the behavior of the exact solution up to the time interval of order $O(\frac{1}{\epsilon^2})$. That is why we expect that the approximate solutions created by the 3-scale method should be uniformly valid on an interval of order $O(\frac{1}{\epsilon^2})$ or longer.

In this thesis, we use the 3-scale method on the periodic first order ordinary differential equation of the form

$$\begin{cases} \frac{dx}{dt} = \epsilon f(t, x) \\ x(0, \epsilon) = \alpha \end{cases}$$

where $x \in R^n$ and $\alpha \neq 0$. We also require that $f(t, x)$ is 2π-periodic in t, and at least a C^2 function in t and the vector x. We will show that the 3-scale method may fail to give an approximate solution on an interval of order $O(\frac{1}{\epsilon^2})$. When we study the usual way of finding a 3-scale approximate solution, we find out that it requires each term after the leading term to be bounded in order to produce a uniformly ordered asymptotic series. But the definition of "uniformly ordered" and "uniform validity" are too strong for some applications. Therefore we will give the weaker definitions of "uniform ordering" and "uniform validity". If an asymptotic series

$$x_0(t, \epsilon)\delta_0(\epsilon) + x_1(t, \epsilon)\delta_1(\epsilon) + \cdots + x_k(t, \epsilon)\delta_k(\epsilon) \tag{1.1}$$

where $\delta_0, \delta_1, \cdots, \delta_k$ are the gauge functions, is uniformly ordered, it means the following.

Definition 1 An asymptotic series (1.1) is uniformly ordered if there are gauge functions $\Delta_0, \Delta_1, \cdots, \Delta_k$ such that $\Delta_i = o(\Delta_{i-1})$ for $i = 1, 2, \cdots, k$ and

$$x_i(t,\epsilon)\delta_i(\epsilon) = O(\Delta_i(\epsilon)$$

for $i = 0, 1, \cdots, k$.

When we say that (1.1) is a uniformly valid approximation of order $\delta(\epsilon)$ for t in an interval I_ϵ, which may depend on ϵ, to $x(t,\epsilon)$ provided that
(i) the (1.1) is a uniform asymptotic expansion for $t \in I_\epsilon$, and
(ii) $\|x(t,\epsilon) - (x_0(t,\epsilon)\delta_0(\epsilon) + \cdots + x_k(t,\epsilon)\delta_k(\epsilon))\| = O(\delta(\epsilon))$
uniformly for $t \in I_\epsilon$.
When we say that (1.1) is a uniformly asymptotic expansion for $t \in I_\epsilon$, we mean the following

Definition 2 The approximation (1.1) to a function $x(t,\epsilon)$ is a uniform asymptotic expansion for $t \in I_\epsilon$ provided that there exist gauge functions $\Delta_1, \Delta_2, \cdots, \Delta_k$ such that $\Delta_i = o(\Delta_{i-1})$ and each of the following statements holds uniformly for $t \in I_\epsilon$

$$\begin{aligned}
x(t,\epsilon) &= x_0(t,\epsilon)\delta_0(\epsilon) + O(\Delta_1(\epsilon)) \\
x(t,\epsilon) &= x_0(t,\epsilon)\delta_0(\epsilon) + x_1(t,\epsilon)\delta_1(\epsilon) + O(\Delta_2(\epsilon)) \\
&\vdots \\
x(t,\epsilon) &= \sum_{i=0}^{i=k} x_i(t,\epsilon)\delta_i(\epsilon) + o(\Delta_k(\epsilon))
\end{aligned}$$

The next theorem will tells us that a uniform asymptotic expansion is uniformly ordered.

Theorem 1 If (1.1) is a uniform asymptotic expansion for $t \in I_\epsilon$ then it is uniformly ordered.

Proof: Since (1.1) is a uniform asymptotic expansion for $t \in I_\epsilon$, we have

$$x(t,\epsilon) - \sum_{i=0}^{i=j-1} x_i(t,\epsilon)\delta_i(\epsilon) \;= O(\Delta_j(\epsilon)) \tag{1.2}$$

$$x(t,\epsilon) - \sum_{i=0}^{i=j} x_i(t,\epsilon)\delta_i(\epsilon) \;= O(\Delta_{j+1}(\epsilon)). \tag{1.3}$$

Subtract (1.2) from (1.3) and we obtain

$$x_j(t,\epsilon)\delta_j(\epsilon) = O(\Delta_j(\epsilon)).$$

therefore (1.1) is uniformly ordered.

Q.E.D.

If one requires $\Delta_i = \delta_i$ in the previous definitions, the usual notions of uniform ordering and uniform validity are recovered, and uniform ordering becomes equivalent to boundedness of $x_i(t,\epsilon)$ for $t \in I_\epsilon$. In this thesis we want to study whether the 3-scale method will produce a uniformly valid approximate solution on an expanding interval of order $O(\frac{1}{\epsilon^2})$ or longer. In chapter 1 we state the structures and many formulas for the 3-scale method and one example which shows us that the 3-scale method can fail. Then we turn to find out under which conditions the 3-scale method will work. In chapter 2 we conclude that if $\int_0^{2\pi} f(t,x)\,dt = 0$ then the 3-scale method always produces a uniformly valid approximate solution on an interval of order $O(\frac{1}{\epsilon^2})$. Furthermore in this case the 3-scale method produces the same approximate solution as the method of averaging. In chapter 3 we want to know whether the 3-scale method has the corresponding result as the Sanchez-Palencia Theorem for the averaging method when a contracting property is present.. The answer is yes. We also show the uniformly validity of higher order approximation.

CHAPTER 2. FORMAL APPROXIMATION USING THREE TIME SCALE

Consider the initial value problem

$$\begin{cases} \dot{x} = \epsilon f(t,x), \quad x \in R^n \\ x(0) = \alpha \end{cases} \tag{2.1}$$

where f is at least C^2 in all its arguments with $f(t+2\pi,x) = f(t,x)$ and ϵ a small nonnegative parameter.

According to the ideas of the 3-scale method, one looks for an approximate solution $y(t,\tau,\sigma)$ of the form

$$\begin{aligned} y(t,\tau,\sigma) &= x_0(t,\tau,\sigma) + \epsilon x_1(t,\tau,\sigma) + \epsilon^2 x_2(t,\tau,\sigma) + \\ &\quad \epsilon^3 x_3(t,\tau,\sigma) + \epsilon^4 x_4(t,\tau,\sigma) + \cdots \end{aligned} \tag{2.2}$$

where the functions $x_i, i = 0,1,2,3,\ldots$, again are required to be 2π-periodic in t, and where $\tau = \epsilon t$ and $\sigma = \epsilon^2 t$ are treated as independent variables, such that $y(t,\epsilon t,\epsilon^2 t)$ is a uniformly valid solution on an interval of order $O(\frac{1}{\epsilon^2})$. Starting from this general idea, a recursive scheme can be developed to determine the unknown functions x_i

order by order. The procedure we will follow is the usual one. First we set up iterative relations for the determination of the x_i's by requiring $y(t,\tau,\sigma)$ to be a formal approximation of the exact solution, $x_{exact}(t,\epsilon)$ of (2.1) to a desired order in ϵ. This means that $y(t,\tau,\sigma)$ has to satisfy formally equation (2.1) to a desired order in ϵ. In order that $y(t,\tau,\sigma)$ be a formal approximation, we require the expression

$$\frac{dy(t,\tau,\sigma)}{dt} - \epsilon f(t, y(t,\tau,\sigma)) \tag{2.3}$$

to vanish to the desired order in ϵ. Up to the fourth order in ϵ we have

$$\begin{aligned}\frac{dy}{dt} &= \frac{\partial x_0(t,\tau,\sigma)}{\partial t} + \epsilon\left[\frac{\partial x_0(t,\tau,\sigma)}{\partial \tau} + \frac{\partial x_1(t,\tau,\sigma)}{\partial t}\right] \\ &\quad + \epsilon^2\left[\frac{\partial x_0(t,\tau,\sigma)}{\partial \sigma} + \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau} + \frac{\partial x_2(t,\tau,\sigma)}{\partial t}\right] \\ &\quad + \epsilon^3\left[\frac{\partial x_1(t,\tau,\sigma)}{\partial \sigma} + \frac{\partial x_2(t,\tau,\sigma)}{\partial \tau} + \frac{\partial x_3(t,\tau,\sigma)}{\partial t}\right] \\ &\quad + \epsilon^4\left[\frac{\partial x_2(t,\tau,\sigma)}{\partial \sigma} + \frac{\partial x_3(t,\tau,\sigma)}{\partial \tau} + \frac{\partial x_4(t,\tau,\sigma)}{\partial t}\right] + \cdots \end{aligned} \tag{2.4}$$

and we also have

$$\begin{aligned}\epsilon f(t,y) &= \epsilon f(t, x_0(t,\tau,\sigma)) + \epsilon^2 f_x(t, x_0(t,\tau,\sigma)) \cdot x_1(t,\tau,\sigma) \\ &\quad + \epsilon^3[f_x(t, x_0(t,\tau,\sigma)) \cdot x_2(t,\tau,\sigma) + \frac{1}{2} f_{xx}(t, x_0(t,\tau,\sigma)) \cdot x_1(t,\tau,\sigma) \cdot x_1(t,\tau,\sigma)] \\ &\quad + \epsilon^4[f_x(t, x_0(t,\tau,\sigma)) \cdot x_3(t,\tau,\sigma) + f_{xx}(t, x_0(t,\tau,\sigma)) \cdot x_1(t,\tau,\sigma) \cdot x_2(t,\tau,\sigma) \\ &\quad + \frac{1}{6} f_{xxx}(t, x_0(t,\tau,\sigma)) \cdot x_0(t,\tau,\sigma) \cdot x_0(t,\tau,\sigma) \cdot x_0(t,\tau,\sigma)] + \cdots \end{aligned} \tag{2.5}$$

where f_x, f_{xx} and f_{xxx} are the first, the second, and the third derivatives of $f(t,x)$ with respect to the vector x. Thus f_x is a matrix, and f_{xx} and f_{xxx} are indexed arrays (formal tensors) of higher rank. The notation $f_{xx} \cdot a \cdot b$ denotes the vector whose i^{th} component is

$$\sum_{j,k} \left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\right) a_j b_k,$$

with a similar formula for $f_{xxx} \cdot a \cdot b \cdot c$. Equations (2.4) and (2.5) must be equal up to the order which we desire. We collect the like terms of the same order of ϵ together and have

$$\begin{cases} \dfrac{\partial x_0(t,\tau,\sigma)}{\partial t} = 0 \\ x_0(0,0,0) = \alpha. \end{cases} \tag{2.6}$$

$$\begin{cases} \dfrac{\partial x_1(t,\tau,\sigma)}{\partial t} = f(t, x_0(t,\tau,\sigma)) - \dfrac{\partial x_0(t,\tau,\sigma)}{\partial \tau} \\ x_1(0,0,0) = 0 \end{cases} \tag{2.7}$$

$$\begin{cases} \dfrac{\partial x_2(t,\tau,\sigma)}{\partial t} = f_x(t, x_0(t,\tau,\sigma)) \cdot x_1(t,\tau,\sigma) - \dfrac{\partial x_0(t,\tau,\sigma)}{\partial \sigma} - \dfrac{\partial x_1(t,\tau,\sigma)}{\partial \tau} \\ x_2(0,0,0) = 0 \end{cases} \tag{2.8}$$

$$\begin{cases} \dfrac{\partial x_3(t,\tau,\sigma)}{\partial t} = f_x(t, x_0(t,\tau,\sigma)) \cdot x_2(t,\tau,\sigma) + \frac{1}{2} f_{xx}(t, x_0(t,\tau,\sigma)) \cdot x_1(t,\tau,\sigma) \cdot x_1(t,\tau,\sigma) \\ \qquad - \dfrac{\partial x_1(t,\tau,\sigma)}{\partial \sigma} - \dfrac{\partial x_2(t,\tau,\sigma)}{\partial \tau} \\ x_3(0,0,0) = 0 \end{cases} \tag{2.9}$$

Equation (2.6) immediately yields $x_0(t,\tau,\sigma)=\hat{x}_0(\tau,\sigma)$ and $\hat{x}_0(0,0) = \alpha$. Thus (2.7), (2.8), and (2.9) become

$$\begin{aligned} \frac{\partial x_1(t,\tau,\sigma)}{\partial t} &= f(t, \hat{x}_0(\tau,\sigma)) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau}, & (2.10) \\ \frac{\partial x_2(t,\tau,\sigma)}{\partial t} &= f_x(t, \hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} - \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau}, & (2.11) \\ \frac{\partial x_3(t,\tau,\sigma)}{\partial t} &= f_x(t, \hat{x}_0(\tau,\sigma)) \cdot x_2(t,\tau,\sigma) \\ &\quad + \frac{1}{2} f_{xx}(t, \hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma) \cdot x_1(t,\tau,\sigma) \\ &\quad - \frac{\partial x_1(t,\tau,\sigma)}{\partial \sigma} - \frac{\partial x_2(t,\tau,\sigma)}{\partial \tau}. & (2.12) \end{aligned}$$

Then keeping in mind that the x_i's are required to be bounded for $t \geq 0$ in an interval of length of order $\frac{1}{\epsilon^2}$, we can integrate equation (2.10) with respect to t to obtain

$$x_1(t,\tau,\sigma) = \hat{x}_1(\tau,\sigma) + \int_0^t \left(f(s,\hat{x}_0(\tau,\sigma)) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} \right) ds \tag{2.13}$$

with $\hat{x}_1(0,0) = 0$. Since $\hat{x}_0(\tau,\sigma)$ and $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau}$ depend only on τ and σ, they are constants during the integration. To eliminate the unbounded term in t of $x_1(t,\tau,\sigma)$ for t in an interval of order $O(\frac{1}{\epsilon^2})$, we require the integral of (2.13) to be bounded which is equivalent to requiring that the mean of $f(t,\hat{x}_0(\tau,\sigma)) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau}$ is zero. That is

$$\frac{1}{2\pi} \int_0^{2\pi} \left[f(t,\hat{x}_0(\tau,\sigma)) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} \right] dt = 0.$$

Therefore we have

$$\begin{cases} \dfrac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} = \bar{f}(\hat{x}_0(\tau,\sigma)) \\ \hat{x}_0(0,0) = \alpha \end{cases} \tag{2.14}$$

where

$$\bar{f}(z) = \frac{1}{2\pi} \int_0^{2\pi} f(t,z)\, dt. \tag{2.15}$$

Let $\tilde{f}(t,z) = f(t,z) - \bar{f}(z)$ which we call the oscillatory part of $f(t,z)$. Then equation (2.13) becomes

$$\begin{aligned} x_1(t,\tau,\sigma) &= \hat{x}_1(\tau,\sigma) + \int_0^t \tilde{f}(s,\hat{x}_0(\tau,\sigma))\, ds \\ &= \hat{x}_1(\tau,\sigma) + f_1(t,\hat{x}_0(\tau,\sigma)) \end{aligned} \tag{2.16}$$

where $\hat{x_1}(0,0) = 0$ and

$$f_1(t,z) = \int_0^t \tilde{f}(s,z)\, ds. \tag{2.17}$$

Note that $f_1(t,z)$ is fully determined without knowing $\hat{x}_0(\tau,\sigma)$. Using these results equation (2.11) can be integrated with respect to t to obtain

$$x_2(t,\tau,\sigma) = \hat{x}_2(\tau,\sigma) + \int_0^t \left[f_x(s,\hat{x}_0(\tau,\sigma)) \cdot x_1(s,\tau,\sigma) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} - \frac{\partial x_1(s,\tau,\sigma)}{\partial \tau} \right] ds \qquad (2.18)$$

where $\hat{x}_2(0,0) = 0$.

For $x_2(t,\tau,\sigma)$ to be bounded for t in an interval of order $O(\frac{1}{\epsilon^2})$, we require that the integrand of (2.18) have zero average over one period in t. This means that we must choose $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$ and $\frac{\partial x_1(t,\tau,\sigma)}{\partial \tau}$ such that the integrand of (2.18)

$$f_x(t,\hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} - \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau} \qquad (2.19)$$

contains only the oscillatory part of $f_x(t,\hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma) - \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau}$, for $\hat{x}_0(\tau,\sigma)$ does not depend on t. That is to require

$$\frac{1}{2\pi} \int_0^{2\pi} \left[f_x(t,\hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} - \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau} \right] dt = 0. \qquad (2.20)$$

Differentiating (2.16) with respect to τ and using (2.14), we have

$$\begin{aligned} \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau} &= \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} + f_{1x}(t,\hat{x}_0(\tau,\sigma)) \cdot \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} \\ &= \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} + f_{1x}(t,\hat{x}_0(\tau,\sigma)) \cdot \bar{f}(\hat{x}_0(\tau,\sigma)). \end{aligned} \qquad (2.21)$$

Before we use (2.21) and (2.16) in (2.20), let us see what $\int_0^{2\pi} \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau}\, dt$ is.

$$\begin{aligned}
\int_0^{2\pi} \frac{\partial x_1(t,\tau,\sigma)}{\partial \tau}\, dt &= \int_0^{2\pi} \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau}\, dt + \int_0^{2\pi} \frac{\partial f_1(t,\hat{x}_0(\tau,\sigma))}{\partial \tau}\, dt \\
&= \int_0^{2\pi} \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau}\, dt + \int_0^{2\pi} f_{1x}(t,\hat{x}_0(\tau,\sigma)) \cdot \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau}\, dt \\
&= \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} 2\pi + \int_0^{2\pi} f_{1x}(t,\hat{x}_0(\tau,\sigma)) \cdot \bar{f}(\hat{x}_0(\tau,\sigma))\, dt \\
&= \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} 2\pi + \int_0^{2\pi} f_{1x}(t,\hat{x}_0(\tau,\sigma))\, dt \cdot \bar{f}(\hat{x}_0(\tau,\sigma)) \\
&= \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} 2\pi + [\bar{f}_{1x}(\hat{x}_0(\tau,\sigma)) \cdot \bar{f}(\hat{x}_0(\tau,\sigma))] 2\pi
\end{aligned}$$

where $f_{1x}(t,z)$ is the derivative of $f_1(t,z)$ with respect to the vector z and $\bar{f}_{1x}(z) = \frac{1}{2\pi}\int_0^{2\pi} f_{1x}(t,z)\, dt$. Also

$$\begin{aligned}
\int_0^{2\pi} f_x(t,\hat{x}_0(\tau,\sigma)) \cdot x_1(t,\tau,\sigma)\, dt &= \int_0^{2\pi} f_x(t,\hat{x}_0(\tau,\sigma)) \cdot \hat{x}_1(\tau,\sigma)\, dt \\
&\quad + \int_0^{2\pi} f_x(t,\hat{x}_0(\tau,\sigma)) \cdot f_1(t,\hat{x}_0(\tau,\sigma))\, dt \\
&= 2\pi \left[\bar{f}_x(t,\hat{x}_0(\tau,\sigma)) \cdot \hat{x}_1(\tau,\sigma) + \rho(\hat{x}_0(\tau,\sigma))\right]
\end{aligned}$$

where $\rho(z) = \frac{1}{2\pi}\int_0^{2\pi} f_x(t,z) \cdot f_1(t,z)\, dt$. Using last two equations and (2.21) and (2.16) in (2.20) we find the equation that $\frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau}$ satisfies.

$$\begin{aligned}
&\frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} - \bar{f}_x(\hat{x}_0(\tau,\sigma)) \cdot \hat{x}_1(\tau,\sigma) \\
&= \rho(\hat{x}_0(\tau,\sigma)) - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} - \bar{f}_{1x}(\hat{x}_0(\tau,\sigma)) \cdot \bar{f}(\hat{x}_0(\tau,\sigma))
\end{aligned} \tag{2.22}$$

where

$$\bar{f}_x(z) = \frac{1}{2\pi}\int_0^{2\pi} f_x(s,z)\, ds$$

and

$$\bar{f}_{1x}(z) = \frac{1}{2\pi}\int_0^{2\pi} f_{1x}(s,z)\, ds. \tag{2.23}$$

Equation (2.22) should be able to help us to determinate $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$ by choosing $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$ such that $\hat{x}_1(\tau,\sigma)$ does not containing unbounded terms for τ in some interval of length of order $\frac{1}{\epsilon}$. Unfortunately, this is not always possible. We will give an example after we have all the formulas which we need.

If we can pick a $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$ such that $\hat{x}_1(\tau,\sigma)$ is bounded for τ in an interval of order $O(\frac{1}{\epsilon})$, then (2.18) becomes

$$x_2(t,\tau,\sigma) = \hat{x}_2(\tau,\sigma) + f_2(t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma)) \tag{2.24}$$

where $\hat{x}_2(0,0) = 0$ and $f_2(t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma))$ is the integral of the oscillatory part of $f_x(t,\hat{x}_0(\tau,\sigma))\cdot x_1(t,\tau,\sigma) - \dfrac{\partial x_1(t,\tau,\sigma)}{\partial \tau}$ with respect to t. Using (2.24) in (2.12) and integrating it with respect to t we find that

$$\begin{aligned} x_3(t,\tau,\sigma) &= \hat{x}_3(\tau,\sigma) + \int_0^t f_x(s,\hat{x}_0(\tau,\sigma))\cdot x_2(s,\tau,\sigma)\,ds \\ &+ \frac{1}{2}\int_0^t f_{xx}(s,\hat{x}_0(\tau,\sigma))\cdot x_1(s,\tau,\sigma)\cdot x_1(s,\tau,\sigma)\,ds \\ &- \int_0^t \left[\frac{\partial x_1(s,\tau,\sigma)}{\partial \sigma} + \frac{\partial x_2(s,\tau,\sigma)}{\partial \tau}\right] ds. \end{aligned} \tag{2.25}$$

To require $x_3(t,\tau,\sigma)$ to be bounded for $t$in an interval of order $\frac{1}{\epsilon^2}$, we need that the integrand of (2.25) contains only the oscillatory part of

$$f_x(s,\hat{x}_0(\tau,\sigma))\cdot x_2(t,\tau,\sigma) + \frac{1}{2}f_{xx}(s,\hat{x}_0(\tau,\sigma))\cdot x_1(t,\tau,\sigma)\cdot x_1(t,\tau,\sigma) - \frac{\partial x_1(t,\tau,\sigma)}{\partial \sigma} - \frac{\partial x_2(t,\tau,\sigma)}{\partial \tau}.$$

This means that we require $\dfrac{\partial \hat{x}_2(\tau,\sigma)}{\partial \tau}$ to satisfy

$$\frac{\partial \hat{x}_2(\tau,\sigma)}{\partial \tau} - \bar{f}_x(\hat{x}_0(\tau,\sigma))\cdot \hat{x}_2(\tau,\sigma) = F_2(t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma)) \tag{2.26}$$

where

$$F_2(t,p,q) = \frac{1}{2\pi}\int_0^{2\pi}\left[\tilde{f}_x(t,p)\cdot \tilde{f}_2(t,p,q) + \frac{1}{2}f_{xx}(s,p)\cdot q\cdot q - \frac{\partial q}{\partial \sigma} - \frac{\partial f_2(t,p,q)}{\partial \tau}\right] ds$$

with $p,q \in R^n$. Note that (2.26) and (2.22) are the same linear differential equation except for the different nonhomogeneous parts.

This procedure can be carried on to any order and in the k^{th} order we have

$$x_k(t,\tau,\sigma) = \hat{x_k}(\tau,\sigma) + f_k(t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma),\cdots,x_{k-1}(t,\tau,\sigma)) \tag{2.27}$$

and $\hat{x_k}(\tau,\sigma)$ satisfies

$$\frac{\partial\hat{x_k}(\tau,\sigma)}{\partial\tau} - \bar{f}_x(\hat{x}_0(\tau,\sigma))\cdot\hat{x_k}(\tau,\sigma) = F_k(t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma),\cdots,x_{k-1}(t,\tau,\sigma)) \tag{2.28}$$

where f_k and F_k are functions in $t,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma),\cdots,x_{k-1}(t,\tau,\sigma)$ which can be found from $f,\hat{x}_0(\tau,\sigma),x_1(t,\tau,\sigma),\cdots,x_{k-1}(t,\tau,\sigma)$. Hence we have a recursive procedure to find each x_i. These formulas are intended to show the structure which will appear in higher order approximations; but for the rest of this chapter we will concentrate our attention on the construction of a zero-order asymptotic approximation, for which we will need the formal results up to the second order.

Formulas in terms of Fourier series

Because we will use the Fourier series in some of the proofs, we will write down some formulas which we have in terms of Fourier series. Since $f(t,x)$ has at least two continuous derivatives, it can be written as a uniformly convergent Fourier series for x in any compact set $K \subseteq R^n$.

$$f(t,x) = \sum_{n=-\infty}^{n=\infty} \mathrm{a}_n(x)e^{int} \qquad x \in K. \tag{2.29}$$

Here the $\mathrm{a}_n(x)$'s are complex vector functions with $\mathrm{a}_n(x) = \mathrm{a}_n{}^*(x)$ for all n, where $*$ denotes complex conjugate. Of course

$$\begin{aligned} \bar{f}(x) &= \frac{1}{2\pi}\int_0^{2\pi}\sum_{n=-\infty}^{n=\infty} \mathrm{a}_n(x)e^{int}\,dt && (2.30)\\ &= \mathrm{a}_0(x) && (2.31)\end{aligned}$$

is real.

If $x_0(t, \epsilon t, \epsilon^2 t)$ is in K, equation (2.14) can be written in terms of Fourier series. Then we have

$$\begin{cases} \dfrac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} = \mathrm{a}_0(\hat{x}_0(\tau,\sigma)) \\ x_0(0,0) = \alpha. \end{cases} \tag{2.32}$$

By (2.17) we have

$$\begin{aligned} f_1(t,z) &= \int_0^t \tilde{f}(s,z)ds \\ &= \int_0^t \sum_{n\neq 0} \mathrm{a}_n(z) e^{ins}\, ds \\ &= \sum_{n\neq 0} \frac{\mathrm{a}_n(z)[e^{int}-1]}{in}. \end{aligned} \tag{2.33}$$

Also (2.16) becomes

$$x_1(t,\tau,\sigma) = \hat{x}_1(\tau,\sigma) + \sum_{n\neq 0} \frac{\mathrm{a}_n(\hat{x}_0(\tau,\sigma))[e^{int}-1]}{in}. \tag{2.34}$$

Let us look carefully what $\rho(\hat{x}_0(\tau,\sigma))$ will be in Fourier series.

$$\begin{aligned} \rho(\hat{x}_0(\tau,\sigma)) &= \frac{1}{2\pi}\int_0^{2\pi} f_x(t,\hat{x}_0(\tau,\sigma)) \cdot f_1(t,\hat{x}_0(\tau,\sigma))\, dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=-\infty}^{n=\infty} \mathrm{a}_n'(\hat{x}_0(\tau,\sigma))e^{int} \cdot \sum_{p\neq 0} \mathrm{a}_p(\hat{x}_0(\tau,\sigma)) \left[\frac{e^{ipt}-1}{ip}\right] dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{a}_0'(\hat{x}_0(\tau,\sigma)) \cdot \sum_{p\neq 0} \mathrm{a}_p(\hat{x}_0(\tau,\sigma)) \left[\frac{e^{ipt}-1}{ip}\right] \, ,dt \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \sum_{n\neq 0} \mathrm{a}_n'(\hat{x}_0(\tau,\sigma))e^{int} \cdot \sum_{p\neq 0} \mathrm{a}_p(\hat{x}_0(\tau,\sigma)) \left[\frac{e^{ipt}-1}{ip}\right] dt \\ &= \sum_{n\neq 0} \frac{\mathrm{a}_n'(\hat{x}_0(\tau,\sigma)) \cdot \mathrm{a}_{-n}(\hat{x}_0(\tau,\sigma))}{-in} \\ &\quad - \mathrm{a}_0'(\hat{x}_0(\tau,\sigma)) \cdot \sum_{n\neq 0} \frac{\mathrm{a}_n(\hat{x}_0(\tau,\sigma))}{in} \end{aligned} \tag{2.35}$$

where $a_n'(z)$ is the derivative of $a_n(z)$ with respect to the vector z. Also we have

$$\bar{f}_{1x}(z) \cdot \bar{f}(z) = -\sum_{n\neq 0} \frac{a_n'(z)}{in} \cdot a_0(z) \tag{2.36}$$

Taking the last two equations, (2.22) will become

$$\begin{aligned} &\frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} - a_0'(\hat{x}_0(\tau,\sigma)) \cdot \hat{x}_1(\tau,\sigma) \\ = &\sum_{n\neq 0} \frac{a_n'(\hat{x}_0(\tau,\sigma)) \cdot a_{-n}(\hat{x}_0(\tau,\sigma))}{-in} - \frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} \\ &- a_0'(\hat{x}_0(\tau,\sigma)) \cdot \sum_{n\neq 0} \frac{a_n(\hat{x}_0(\tau,\sigma))}{in} + \sum_{n\neq 0} \frac{a_n'(\hat{x}_0(\tau,\sigma))}{in} \cdot a_0(\hat{x}_0(\tau,\sigma)). \end{aligned} \tag{2.37}$$

These are the formulas which we need for this work.

Example

The following is an example which shows that in general we can not expect to choose $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$ such that $x_1(t,\tau,\sigma)$ will be bounded for $t \sim O\left(\frac{1}{\epsilon^2}\right)$ and $\tau \sim O\left(\frac{1}{\epsilon}\right)$. Consider

$$\dot{x} = \epsilon\left(-x^3 + x^p \cos t + \sin t\right), \qquad x \in R^1 \tag{2.38}$$

$$x(0) = \alpha, \qquad \alpha > 0. \tag{2.39}$$

The best way of doing this example is to write down all the functions that we need for computation. Thus we have

$$\begin{aligned} f(x,t) &= -x^3 + x^p \cos t + \sin t \\ \bar{f}(x,t) &= -x^3 \\ \tilde{f}(x,t) &= x^p \cos t + \sin t \end{aligned}$$

$$\begin{aligned}
f_1(x,t) &= x^p \sin t - \cos t + 1 \\
f_x(x,t) &= -3x^2 + px^{p-1}\cos t \\
\rho(x) &= -3x^2 - \frac{1}{2}px^{p-1} \\
f_{1x}(x,t) &= px^{p-1}\sin t \\
\bar{f}_{1x}(x) &= 0.
\end{aligned}$$

According to (2.6), we have

$$\frac{\partial x_0(t,\tau,\sigma)}{\partial t} = 0, \qquad x_0(0,0,0) = \alpha. \tag{2.40}$$

This implies that

$$x_0(t,\tau,\sigma) = \hat{x}_0(\tau,\sigma), \qquad \hat{x}_0(0,0) = \alpha. \tag{2.41}$$

Using (2.14) we find that

$$\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} = -\left[\hat{x}_0(\tau,\sigma)\right]^3, \qquad \hat{x}_0(0,0) = \alpha. \tag{2.42}$$

This is solvable and we obtain

$$\hat{x}_0(\tau,\sigma) = \left[2\tau + \breve{x}_0(\sigma)\right]^{-1/2}, \quad \breve{x}_0(0) = \alpha^{-2}. \tag{2.43}$$

Then by (2.16) we have

$$x_1(t,\tau,\sigma) = \hat{x}_1(\tau,\sigma) + [\hat{x}_0(\tau,\sigma)]^p \sin t - \cos t + 1. \tag{2.44}$$

Hence

$$\frac{\partial x_1(t,\tau,\sigma)}{\partial \tau} = \frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} - p\left[\hat{x}_0(\tau,\sigma)\right]^{p+2}\sin t. \tag{2.45}$$

Now using (2.44) then (2.22) becomes

$$\begin{aligned}
\frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} + 3\left[\hat{x}_0(\tau,\sigma)\right]^2 \cdot \hat{x}_1(\tau,\sigma) &= -3\left[\hat{x}_0(\tau,\sigma)\right]^2 - \frac{p}{2}\left[\hat{x}_0(\tau,\sigma)\right]^{p-1} \\
&+ \frac{1}{2}[\hat{x}_0(\tau,\sigma)]^3 \frac{\partial \breve{x}_0(\sigma)}{\partial \sigma}
\end{aligned} \tag{2.46}$$

Equation (2.46) is solvable for $\hat{x}_1(\tau,\sigma)$, and it has solution

$$\hat{x}_1(\tau,\sigma) = -1 - \frac{p}{2(6-p)}[2\tau + \breve{x}_0(\sigma)]^{\frac{3-p}{2}} + \left[\frac{\tau}{2[2\tau + \breve{x}_0(\sigma)]^{\frac{3}{2}}}\right]\frac{d\breve{x}_0(\sigma)}{d\sigma} + \breve{x}_1(\sigma) \quad (2.47)$$

where $\breve{x}_1(0) = \alpha^{-3} + \dfrac{p\alpha^{p-6}}{2(6-p)}$.

If $0 < p < 3$, then by (2.47) $\hat{x}_1(\tau,\sigma)$ will be of order $\left(\frac{1}{\epsilon}\right)^{\frac{3-p}{2}}$, where τ is in an interval of length of order $\frac{1}{\epsilon}$, which is unbounded as $\epsilon \to 0$. Therefore there is no way we can pick $\frac{d\breve{x}_0(\sigma)}{d\sigma}$ to stop $\hat{x}_1(\tau,\sigma)$ from becoming unbounded. Hence it is no longer the case that $\epsilon x_1(t,\tau,\sigma)$ is a small correction to $\breve{x}_0(\sigma)$, and the method fails. If we pick $p = 2$ then

$$\hat{x}_1(\tau,\sigma) = -1 - \frac{p}{2(6-p)}[2\tau + \breve{x}_0(\sigma)]^{-\frac{1}{2}} + \left\{\frac{\tau}{2[2\tau + \breve{x}_0(\sigma)]^{\frac{3}{2}}}\right\}\frac{d\breve{x}_0(\sigma)}{d\sigma} + \breve{x}_1(\sigma).$$

Hence $\epsilon x_1(t,\tau,\sigma) = \epsilon\hat{x}_1(\tau,\sigma) + \epsilon f_1(t,\breve{x}_0(\sigma))$ which will be of order $O(\epsilon^{\frac{1}{2}})$ for τ in an interval of order $O(\frac{1}{\epsilon})$. Then $\breve{x}_0(\sigma) + \epsilon x_1(t,\tau,\sigma)$ is not uniformly ordered for the usual sense but $x_0(\sigma) + \epsilon x_1(t,\tau,\sigma)$ is uniformly ordered by our definition for $\epsilon x_1(t,\epsilon t,\epsilon^2 t) = O(\epsilon^{\frac{1}{2}})$ on an expanding interval of order $O(\epsilon^{-2})$. Therefore $\breve{x}_0(\sigma) + \epsilon x_1(t,\tau,\sigma)$ may still be a uniformly valid approximation to $x_{exact}(t,\epsilon)$.

CHAPTER 3. ASYMPTOTIC VALIDITY IF $f(t,x)$ HAS ZERO MEAN

Let us consider the special case when

$$\frac{1}{2\pi}\int_0^{2\pi} f(t,x)\,dt = 0 \tag{3.1}$$

for fixed x. For this special case we will use the Fourier series to show that the 3-scale method works well. Assume that $\hat{x}_0(\epsilon t, \epsilon^2 t)$ is bounded for t in an interval of order $O(\frac{1}{\epsilon^2})$. Then $\hat{x}_0(\epsilon t, \epsilon^2 t)$ is contained in some compact set K in R^n and because $f(t,z)$ is at least C^2 hence $f(t,z)$ has a uniformly convergent Fourier series for z in K. That is

$$f(t,z) = \sum_{n=-\infty}^{n=\infty} \mathrm{a}_n(z)e^{int}, \qquad z \in K \tag{3.2}$$

with $\mathrm{a}_0(z) = 0$. We will modify the formulas for this case.

Equation (2.14) now becomes

$$\begin{cases} \dfrac{\partial \hat{x}_0(\tau,\sigma)}{\partial \tau} = 0 \\ \hat{x}_0(0,0) = \alpha. \end{cases} \tag{3.3}$$

since $\frac{1}{2\pi}\int_0^{2\pi} f(t,z)\,dt = 0$. This implies that

$$\begin{cases} \hat{x}_0(\tau,\sigma) = \breve{x}_0(\sigma) \\ \breve{x}_0(0) = \alpha. \end{cases} \tag{3.4}$$

According to (2.34), $x_1(t,\tau,\sigma)$ will be

$$x_1(t,\tau,\sigma) = \hat{x}_1(\tau,\sigma) + \sum_{n\neq 0} \frac{\mathrm{a_n}(\breve{x}_0(\sigma))\,[e^{int}-1]}{in} \tag{3.5}$$

$$\hat{x}_1(0,0) = 0. \tag{3.6}$$

Using these results equation (2.38) gives us

$$\frac{\partial \hat{x}_1(\tau,\sigma)}{\partial \tau} = \sum_{n\neq 0} \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\sigma)) \cdot \mathrm{a_{-n}}(\breve{x}_0(\sigma))}{-in} - \frac{\partial \breve{x}_0(\sigma)}{\partial \sigma} \tag{3.7}$$

for $\mathrm{a}_0(z) = 0$ hence $\mathrm{a}'_0(z) = 0$.

The first thing to do is to show that if $\breve{x}_0(\sigma)$ is bounded for $t \sim O\left(\frac{1}{\epsilon^2}\right)$, then the x_i's, for $i = 1, 2, 3, \ldots$, are all bounded, hence we have a uniformly ordered sequence. Then we will claim that $\breve{x}_0(\sigma)$ is a uniform approximation to the exact solution, $x_{exact}(t,\epsilon)$ of (2.1). Since $\breve{x}_0(\sigma)$ depends only on σ, the right-hand-side of (3.7) is independent of τ. Therefore we can integrate (3.7) with respect to τ to obtain

$$\hat{x}_1(\tau,\sigma) = \breve{x}_1(\sigma) + \tau\Big[\sum_{n\neq 0} \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\sigma)) \cdot \mathrm{a_{-n}}(\breve{x}_0(\sigma))}{-in} - \frac{\partial \breve{x}_0(\sigma)}{\partial \sigma}\Big]. \tag{3.8}$$

To require $x_1(t,\tau,\sigma)$ to be bounded for τ in an interval of length of order $\frac{1}{\epsilon}$ when $\epsilon \to 0$, we require that the right hand side of (3.7) is zero, that is

$$\begin{cases} \dfrac{\partial \breve{x}_0(\sigma)}{\partial \sigma} = \sum_{n\neq 0} \dfrac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\sigma)) \cdot \mathrm{a_{-n}}(\breve{x}_0(\sigma))}{-in} \\ \breve{x}_0(0) = \alpha \end{cases} \tag{3.9}$$

Equation (3.9) can help us to determine $\breve{x}_0(\sigma)$. Also (3.8) becomes $\hat{x}_1(\tau,\sigma) = \breve{x}_1(\sigma)$. Then (3.5) becomes

$$x_1(t,\tau,\sigma) = \breve{x}_1(\sigma) + \sum_{n\neq 0} \frac{(e^{int}-1)}{in} \mathrm{a_n}(\breve{x}_0(\sigma)) \tag{3.10}$$

and $\breve{x}_1(0) = 0$. This means that x_1 is independent of τ. Hence x_1 is really $x_1(t, \sigma)$, and (3.10) also tells us that $x_1(t, \epsilon^2 t)$ is bounded for $t \sim O(\frac{1}{\epsilon^2})$. In this zero mean case (2.11) becomes

$$\frac{\partial x_2(t, \tau, \sigma)}{\partial t} = f_x(t, \breve{x}_0(\sigma)) \cdot x_1(t, \sigma) - \frac{\partial \breve{x}_0(\sigma)}{\partial \sigma} \tag{3.11}$$

for x_1 does not depend on τ. Then (2.18) becomes

$$\begin{aligned} x_2(t, \tau, \sigma) &= \hat{x}_2(\tau, \sigma) + \\ & \int_0^t \left[f_x(s, \breve{x}_0(\sigma)) \cdot x_1(s, \sigma) - \frac{\partial \breve{x}_0(\sigma)}{\partial \sigma} \right] ds \end{aligned} \tag{3.12}$$

where by (2.26) $\hat{x}_2(\tau, \sigma)$ satisfies

$$\frac{\partial \hat{x}_2(\tau, \sigma)}{\partial \tau} = F_2(t, \breve{x}_0(\sigma), x_1(t, \sigma)). \tag{3.13}$$

Note that $F_2(t, \breve{x}_0(\sigma), x_1(t, \sigma))$ does not depend on τ, so we can integrate the last equation with respect to τ. The only way which can make $\hat{x}_2(\tau, \sigma)$ bounded is that $F_2(t, \breve{x}_0(\sigma), x_1(t, \sigma))$ is zero. This is why we can put down the conditions determining $\frac{\partial \breve{x}_0(\sigma)}{\partial \sigma}$.

In general, for x_j's, for $j = 1, 2, 3, \cdots$, we will have the equations

$$\frac{\partial x_j(t, \tau, \sigma)}{\partial t} = G_j(t, x_0, x_1, \cdots, x_{j-1}) \tag{3.14}$$

$$\frac{\partial \hat{x}_j(\tau, \sigma)}{\partial \tau} = F_j(t, x_0, x_1, \cdots, x_{j-1}). \tag{3.15}$$

Here $x_0, x_1, \cdots, x_{j-1}$ are independent of τ. Therefore we can integrate (3.15) with respect to τ. Again the only way to make x_j bounded is that $F_j(t, x_0, x_1, \cdots, x_{j-1})$ is zero. This means that x_j can be made bounded as long as x_0 is bounded for

$t \sim O(\frac{1}{\epsilon^2})$. Thus we know that x_j's are uniformly ordered and x_j's are functions of t and σ.

Before we prove the asymptotic convergence of this special case, we need the following modified version of a lemma due to Besjes [2].

LEMMA 1 Suppose that $x(t,\epsilon)$ is a solution of the equation

$$\frac{dx(t,\epsilon)}{dt} = \epsilon^r g(t, x(t,\epsilon)) \tag{3.16}$$

$$x(0,\epsilon) = \alpha \tag{3.17}$$

where $r > 0$ and $g(t,z)$ is a 2π-periodic function in t and for each fixed z,

$$\int_0^{2\pi} g(t,z)\,dt = 0. \tag{3.18}$$

Suppose that there exist a constant $\epsilon_0 > 0$ and a compact set K such that $x(t,\epsilon)$ is in K for ϵ in $[0,\epsilon_0]$ and for t in $[0, d(\epsilon)]$ where $d(\epsilon)$ is a positive function of ϵ.
Let $\phi(t,z)$ be a smooth (C^∞) function, 2π-periodic in t , for each fixed z, and

$$\int_0^{2\pi} \phi(t,z)\,dt = 0. \tag{3.19}$$

Then there are constants c_0 and c_1 such that

$$\|\int_0^t \phi(s, x(s,\epsilon))\,ds\| \le c_0 + \epsilon^r c_1 t \tag{3.20}$$

for t in $[0, d(\epsilon)]$, where $d(\epsilon)$ is of order $O(\frac{1}{\epsilon^2})$, and for ϵ in $[0,\epsilon_0]$.
Proof: Since $\phi(t,z)$ is 2π -periodic with zero mean in t and has at least two continuous derivatives, it can be written as a uniformly convergent Fourier series

$$\phi(t,z) = \sum_{n\neq 0} a_n(z) e^{int} \tag{3.21}$$

where the $a_n(z)$'s are vector-valued coefficients satisfying

$$\|a_n(z)\| \leq \frac{k}{n^2} \tag{3.22}$$

for some constant k and for all z in a compact set K. Also

$$\|a_n'(z)\| \leq \frac{l}{n}, \tag{3.23}$$

where a_n' is the matrix of partial derivatives of a_n with respect to z, for some constant l and all z in the compact set K. Now substitute $x(t,\epsilon)$ in (3.19). Since the convergence is uniform, the resulting series can be integrated termwise by parts to obtain

$$\begin{aligned}\int_0^t \phi(s,x(s,\epsilon))\,ds &= \sum_{n\neq 0} \frac{a_n(x(t,\epsilon))e^{int} - a_n(x(0,\epsilon))}{in} \\ &\quad - \sum_{n\neq 0}\int_0^t \frac{e^{ins}a_n'(x(s,\epsilon))}{in}\frac{dx(s,\epsilon)}{ds}\,ds \\ &= \sum_{n\neq 0} \frac{a_n(x(t,\epsilon))e^{int} - a_n(x(0,\epsilon))}{in} \\ &\quad - \sum_{n\neq 0}\int_0^t \epsilon^r \frac{e^{ins}a_n'(x(s,\epsilon))}{in} g(s,x(s,\epsilon))\,ds \end{aligned} \tag{3.24}$$

for $\epsilon \in [0, \epsilon_0]$.

As long as $x(t,\epsilon) \in K$ we can find

$$M_0 = Max\,\{\, \|g(s,z)\| : z \in K \text{ and } s \in [0, 2\pi]\}\,. \tag{3.25}$$

Thus by (3.24), as long as $x(t,\epsilon)$ is in K, we obtain

$$\begin{aligned}\left\|\int_0^t \phi(s,x(s,\epsilon))\,ds\right\| &\leq \sum_{n\neq 0}\frac{2k}{n^3} + \sum_{n\neq 0}\int_0^t \frac{\epsilon^r l M_0}{n^2}\,ds \\ &= \sum_{n\neq 0}\frac{2k}{n^3} + \sum_{n\neq 0}\frac{lM_0\epsilon^r t}{n^2}\end{aligned} \tag{3.26}$$

for $\epsilon \in [0, \epsilon_0]$ and for t in $[0, d(\epsilon)]$. Let $c_0 = \sum_{n \neq 0} \frac{2k}{n^3}$, and $c_1 = \sum_{n \neq 0} \frac{lM_0}{n^2}$. Then we have shown

$$\| \int_0^t \phi(s, x(s, \epsilon))\, ds \| \leq c_0 + c_1 \epsilon^r t \tag{3.27}$$

as long as $x(t, \epsilon)$ in K.

Q.E.D.

It is clear that if $r = 2$ then for an arbitrary positive constant L, we have

$$\| \int_0^t \phi(s, x(s, \epsilon))\, ds \| \leq c_0 + c_1 L$$

for t in $[0, \frac{L}{\epsilon^2}]$ and ϵ in $[0, \epsilon_0]$.

Next is the theorem which states that if $\int_0^{2\pi} f(s, z)\, ds = 0$, then

$$x_{exact}(t, \epsilon) = \breve{x}_0(\sigma) + O(\epsilon),$$

for t in an expanding interval of order $O(\frac{1}{\epsilon^2})$.

THEOREM 2 Let $\breve{x}_0(\sigma)$ be the solution of (3.9) which is used to be an approximate solution to the solution $x_{exact}(t, \epsilon)$ of (2.1). Then there exist positive constants C, ϵ_0 and L such that $\breve{x}_0(\sigma)$ satisfies

$$\|x_{exact}(t, \epsilon) - \breve{x}_0(\sigma)\| < C\epsilon \tag{3.28}$$

for $t \in \left[0, \frac{L}{\epsilon^2}\right]$ and $\epsilon \in [0, \epsilon_0]$.

Proof: First we must show that $\breve{x}_0(\sigma)$ is in some compact set K for $t \in \left[0, \frac{L}{\epsilon^2}\right]$.

Let K be a closed ball centered at α with radius $\delta > 0$. Pick $\epsilon_0 > 0$ such that

$\epsilon_0 \in (0,1)$. From (3.9) we have

$$\begin{cases} \dfrac{d\breve{x}_0(\epsilon^2 t)}{dt} = \epsilon^2 \sum_{n\neq 0} \dfrac{\mathrm{a}'_{\mathrm{n}}(\breve{x}_0(\epsilon^2 t)) \cdot \mathrm{a}_{-\mathrm{n}}(x_0(\epsilon^2 t)}{-in} \\ \breve{x}_0(0) = \alpha. \end{cases} \tag{3.29}$$

By (3.29) and for t in some interval $[0, d(\epsilon)]$, we find

$$\begin{aligned} \|x_0(\epsilon^2 t)\| &\leq \epsilon^2 \int_0^t \left\| \sum_{n\neq 0} \frac{\mathrm{a}'_{\mathrm{n}}(x_0(\epsilon^2 s)) \cdot \mathrm{a}_{-\mathrm{n}}(x_0(\epsilon^2 s))}{-in} \right\| ds \\ &\leq \epsilon^2 M_1 t. \end{aligned} \tag{3.30}$$

where

$$M_1 = Max \left\{ \| \sum_{n\neq 0} \frac{\mathrm{a}'_{\mathrm{n}}(z) \cdot \mathrm{a}_{-\mathrm{n}}(x_0(z))}{-in} \| : z \in K \right\}.$$

Hence for $t \in [0, \frac{L}{\epsilon^2}]$ and $\epsilon \in [0, \epsilon_0]$, from (3.30) we require

$$L \leq \frac{\delta}{M_1 \epsilon^2}$$

to make $x_0(\epsilon^2 t) \in K$ for $t \in [0, \frac{\delta}{M_1 \epsilon^2}]$ and $\epsilon \in [0, \epsilon_0]$. Therefore let $L = \frac{\delta}{M_1 \epsilon^2}$. Next we show $x_0(\epsilon^2 t)$ satisfies (3.28).

Let

$$x_{exact}(t, \epsilon) = \breve{x}_0(\epsilon^2 t) + R(t, \epsilon). \tag{3.31}$$

Then $R(t, \epsilon)$ will satisfy

$$\begin{cases} \dfrac{dR(t,\epsilon)}{dt} = \epsilon f(t, \breve{x}_0(\epsilon^2 t) + R(t, \epsilon)) - \epsilon^2 \sum_{n\neq 0} \dfrac{\mathrm{a}'_{\mathrm{n}}(\breve{x}_0(\epsilon^2 t)) \cdot \mathrm{a}_{-\mathrm{n}}(\breve{x}_0(\epsilon^2 t))}{-in} \\ R(0, \epsilon) = 0 \end{cases} \tag{3.32}$$

for $\epsilon \in [0, \epsilon_0]$. Since $R(0, \epsilon) = 0$, there is some positive length time interval I beginning at $t = 0$ such that $x_0(\epsilon^2 t) + R(t, \epsilon)$ in K for $\epsilon \in [0, \epsilon_0]$ and t in the interval I.

For t in the time interval I we can apply integration by parts and get

$$\begin{aligned}
R(t,\epsilon) &= \epsilon\sum_{n\neq 0}\frac{\mathrm{a_n}(\breve{x}_0(\epsilon^2 t)) + R(t,\epsilon))e^{int} - \mathrm{a_n}(\alpha)}{in} - \epsilon^2\sum_{n\neq 0}\int_0^t \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s))\mathrm{a_{-n}}(\breve{x}_0(\epsilon^2 s))}{-in}\,ds \\
&\quad - \epsilon\sum_{n\neq 0}\int_0^t \frac{e^{ins}}{in}\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\frac{d(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))}{ds}\,ds \\
&= \epsilon\sum_{n\neq 0}\frac{\mathrm{a_n}(\breve{x}_0(\epsilon^2 t) + R(t,\epsilon))e^{int} - \mathrm{a_n}(\alpha)}{in} - \epsilon^2\sum_{n\neq 0}\int_0^t \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s))\cdot \mathrm{a_{-n}}(\breve{x}_0(\epsilon^2 s))}{in}\,ds \\
&\quad - \epsilon^2\sum_{n\neq 0}\int_0^t \frac{e^{ins}}{in}\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\cdot f(s,\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\,ds \qquad (3.33)
\end{aligned}$$

for $x(t,\epsilon) = \breve{x}_0(\epsilon^2 t) + R(t,\epsilon)$ and

$$\frac{dx_{exact}(t,\epsilon)}{dt} = \frac{d[\breve{x}_0(\epsilon^2 t) + R(t,\epsilon)]}{dt} = \epsilon f(t, x_0(\epsilon^2 t) + R(t,\epsilon)).$$

Now let us examine the second ϵ^2 terms in (3.33). Because

$$f(s,z) = \sum_{n\neq 0}\mathrm{a_n}(z)e^{ins}$$

and

$$f_1(s,z) = \sum_{n\neq 0}\frac{[e^{ins}-1]}{in}\mathrm{a_n}(z),$$

we find out that

$$\begin{aligned}
&\sum_{n\neq 0}\int_0^t \frac{e^{ins}}{in}\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\cdot f(s,\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\,ds \\
= &\sum_{n\neq 0}\int_0^t \frac{e^{ins}}{in}\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\cdot\sum_{p\neq 0}\mathrm{a_p}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))e^{ips}\,ds \\
= &\sum_{n+p\neq 0}^{p,n\neq 0}\int_0^t \frac{e^{i(n+p)s}}{in}\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\cdot\mathrm{a_p}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\,ds \\
&+\sum_{n\neq 0}\int_0^t \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\cdot\mathrm{a_{-n}}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))}{in}\,ds. \qquad (3.34)
\end{aligned}$$

Use (3.34) in (3.33) we obtain

$$R(t,\epsilon) = \epsilon \sum_{n\neq 0} \frac{\mathrm{a_n}(\breve{x}_0(\epsilon^2 t) + R(t,\epsilon))e^{int} - \mathrm{a_n}(\alpha)}{in} \tag{3.35}$$
$$- \epsilon^2 \sum_{n,p\neq 0}^{n+p\neq 0} \int_0^t \frac{e^{i(n+p)s}}{in} \mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon)) \cdot \mathrm{a_p}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))\, ds$$
$$+ \epsilon^2 \int_0^t \sum_{n\neq 0} \frac{\mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s)) \cdot \mathrm{a_{-n}}(\breve{\mathrm{x}}_0(\epsilon^2 \mathrm{s})) - \mathrm{a}'_\mathrm{n}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon)) \cdot \mathrm{a_{-n}}(\breve{x}_0(\epsilon^2 s) + R(s,\epsilon))}{in}\, ds.$$

Let $\delta_1 > 0$ such that

$$\|R(t,\epsilon)\| \leq \delta_1 \tag{3.36}$$

for t in the interval I and $\epsilon \in [0, \epsilon_0]$. As long as $\|R(t,\epsilon)\| \leq \delta_1$ we have $\breve{x}_0(\sigma) + R(t,\epsilon)$ in K. (We may have to change δ to $\delta + \delta_1$.) Then (3.35) holds as long as $\breve{x}_0 + R$ is in K and we can apply Lemma 1 to the second term on the right hand side of (3.35). Hence we have

$$\|R(t,\epsilon)\| \leq \epsilon \sum_{n\neq 0} \frac{2k}{n^3} + \epsilon^2(c_0 + c_1 \epsilon t) + \epsilon^2 \int_0^t \|h'(z_0)\| \|R(s,\epsilon)\|\, ds \tag{3.37}$$

where for some z_0 in K and where

$$h(z) = \sum_{n\neq 0} \frac{\mathrm{a}'_\mathrm{n}(z) \cdot \mathrm{a_{-n}}(z)}{in}$$

which has at least one continuous derivatives with respect to the vector z.
Let $M_2 = Max\,\{\|h'(z)\| : z \in K\}$. Then (3.37) becomes

$$\|R(t,\epsilon)\| \leq \epsilon c_0 + \epsilon^2(c_0 + c_1 \epsilon t) + \epsilon^2 \int_0^t M_2 \|R(s,\epsilon)\|\, ds. \tag{3.38}$$

By Gronwall's inequality we obtain that $\|R(t,\epsilon)\|$ satisfies

$$\|R(t,\epsilon)\| \leq \epsilon C(\epsilon) \tag{3.39}$$

as long as $\breve{x}_0 + R$ is in K, and

$$C(\epsilon) = \sum_{n\neq 0} \frac{2k}{n^3} + \epsilon c_0 + \frac{2c_1\delta}{M_1} + (\sum_{n\neq 0} \frac{2k}{n^3} + \epsilon c_0 + \frac{c_1}{M_2})e^{\frac{\delta M_2}{M_1}}.$$

Pick $\epsilon_0 > 0$ such that

$$\epsilon_0 C(\epsilon_0) < \delta_1. \tag{3.40}$$

Then $\breve{x}_0 + R$ is in K for at least $t \in [0, \frac{\delta}{M_1\epsilon^2}]$ and $\epsilon \in [0, \epsilon_0]$. Let $C = C(\epsilon_0)$. Hence for $t \in [0, \frac{\delta}{M_1\epsilon^2}]$ and $\epsilon \in [0, \epsilon_0]$ we have $\|R(t,\epsilon)\| \leq \epsilon C$. That is

$$\|x_{exact}(t,\epsilon) - \breve{x}_0(\epsilon^2 t)\| \leq \epsilon C$$

for $t \in [0, \frac{L}{\epsilon^2}]$, where $L = \frac{\delta}{M_1\epsilon^2}$, and $\epsilon \in [0, \epsilon_0]$.

Q.E.D.

Now we will give an alternative proof of the last theorem which we get by reducing it to the method of averaging. We need to set up the averaging system first.

The exact system is

$$\begin{cases} \frac{dx}{dt} = \epsilon f(t,x) \\ x(0,\epsilon) = \alpha. \end{cases}$$

We seek a transformation of the form

$$x = y + \epsilon u_1(t,y) + \epsilon^2 u_2(t,y)$$

in which u_1 and u_2 is 2π-periodic in t and which carry (2.1) into

$$\frac{dy}{dt} = \epsilon g_1(y) + \epsilon^2 g_2(y) \qquad y(0,\epsilon) = \alpha,$$

where $g_1(y)$ and $g_2(y)$ are independent of t. This will be accomplished provided u_1 and u_2 are chosen so as to satisfy

$$\begin{aligned}\frac{\partial u_1(t,y)}{\partial t} &= f(t,y) - g_1(y) && (3.41)\\ \frac{\partial u_2(t,y)}{\partial t} &= f_x(t,y)u_1(t,y) - u_{1x}(t,y)g_1(y) - g_2(y). && (3.42)\end{aligned}$$

These solutions are 2π-periodic if and only if g_1 and g_2 satisfy the following equations.

$$\begin{aligned}g_1(y) &= \frac{1}{2\pi}\int_0^{2\pi} f(t,y)\,dt\\ &= \bar{f}(y) = 0 && (3.43)\\ g_2(y) &= \frac{1}{2\pi}\int_0^{2\pi}\left\{f_x(t,y)u_1(t,y) - u_{1x}(t,y)g_1(y)\right\}\,dt. && (3.44)\end{aligned}$$

Because u_1 satisfies (3.41) and $g_1(y) = \bar{f}(y) = 0$, u_1 can be solved and we get

$$u_1(t,y) = \int_0^t \tilde{f}(s,y)\,ds. \tag{3.45}$$

Note that according to (2.17) $u_1(t,y) = f_1(t,y)$. Therefore

$$\begin{aligned}g_2(y) &= \frac{1}{2\pi}\int_0^{2\pi}\left[f_x(t,y)f_1(t,y) - f_{1x}(t,y)\bar{f}(y)\right]\,dt\\ &= \frac{1}{2\pi}\int_0^{2\pi} f_x(t,y)f_1(t,y)\,dt\end{aligned}$$

which is exactly the right hand side of (3.9) if we rewrite it in terms of the Fourier series. Now we can put things together and see what we have. In the averaging method we use the solution of the equation

$$\frac{dy}{dt} = \epsilon g_1(y) + \epsilon^2 g_2(y) \tag{3.46}$$

to construct $y + \epsilon u_1 + \epsilon^2 u_2$ as an approximate solution to $x_{exact}(t,\epsilon)$. In this case we have $g_1(y) = \bar{f}(y) = 0$. Thus (3.46) becomes

$$\frac{dy}{dt} = \epsilon^2 g_2(y)$$

which is same as (3.9). Therefore $y = \breve{x}_0$. There is a known result [1; Theorem 6.5.3, p.319] for the method of averaging which states that if $g_1(y) = 0$ then there are constants c and ϵ_1 such that

$$\|x_{exact}(t,\epsilon) - \tilde{x}(t,\epsilon)\| < c\epsilon \tag{3.47}$$

where $\tilde{x}(t,\epsilon) = t + \epsilon u_1(t,y) + \epsilon^2 u_2(t,\epsilon)$, for t in an expanding interval of order $O(\frac{1}{\epsilon^2})$ and ϵ in $[0,\epsilon_1]$. Since $y = \breve{x}_0$ which is bounded in a compact ball K for an interval of order $O(\frac{1}{\epsilon^2})$ and u_1 and u_2 are 2π-periodic in t we know that $\epsilon u_1(t,y) + \epsilon^2 u_2(t,y)$ is of order $O(\epsilon)$ for t in an interval of order $O(\frac{1}{\epsilon^2})$ and ϵ in $[0,\epsilon_1]$. Therefore we have

$$\begin{aligned} \|x(t,\epsilon)_{exact} - \breve{x}_0\| &\leq \|x_{exact} - \tilde{x}\| + \|\tilde{x} - \breve{x}_0\| \\ &\leq \epsilon c + \|\epsilon u_1 + \epsilon^2 u_2\|. \end{aligned} \tag{3.48}$$

Hence there are positive constants ϵ_1 ,c_3 such that

$$\|x_{exact}(t,\epsilon) - \breve{x}_0(\epsilon^2 t)\| \leq c_3\epsilon \tag{3.49}$$

for t in an interval of order $O(\frac{1}{\epsilon^2})$ and for ϵ in $[0,\epsilon_1]$.

CHAPTER 4. THE CASE $\int_0^{2\pi} f(t,x)\,dt \neq 0$

There are two things affecting whether $y(t,\tau,\sigma)$ is a uniformly valid approximate solution to $x_{exact}(t,\epsilon)$. First is the boundedness of $\hat{x}_0(\epsilon t, \epsilon^2 t)$ on an expanding interval of order $O(\frac{1}{\epsilon^2})$. If $\hat{x}_0(\epsilon t, \epsilon^2 t)$ is not bounded on an interval of order $O(\frac{1}{\epsilon^2})$, then according to (2.22) $x_1(t, \epsilon t, \epsilon^2 t)$ will be affected by $\hat{x}_0(\epsilon t, \epsilon^2 t)$ through $\hat{x}_1(\epsilon t, \epsilon^2 t)$. Thus $x_1(t, \epsilon t, \epsilon^2 t)$ may be unbounded. If $\hat{x}_0$ and x_1 are unbounded then $\hat{x}_2$ will be affected by (2.26) and it may be unbounded on an interval of order $O(\frac{1}{\epsilon^2})$. This situation usually will go to higher order terms to cause them to be unbounded on an interval of order $O(\frac{1}{\epsilon^2})$. Therefore the higher order term may be no longer a small correction to the one order lower term. Second if the ordinary differential equation

$$\frac{dz}{d\tau} - \bar{f}_x(\hat{x}_0(\tau,\sigma)) \cdot z = 0$$

has some of its solutions to be unbounded on an expanding time interval of order $O(\frac{1}{\epsilon^2})$, then $\hat{x}_i(\epsilon t, \epsilon^2 t)$, for $i = 1, 2, 3, \cdots$, will unbounded on the same interval even when $\hat{x}_0(\epsilon t, \epsilon^2 t)$ is bounded. Therefore we will give two assumptions:

(a) there exists an $\epsilon_1 > 0$ such that $\hat{x}_0(\epsilon t, \epsilon^2 t)$ is bounded for t in an expanding interval of order $O(\frac{1}{\epsilon^2})$ and for ϵ in $[0, \epsilon_1]$. Hence $\hat{x}_0(\epsilon t, \epsilon^2 t)$ will be contained in a compact ball K centered at α for t in an expanding interval of order $O(\frac{1}{\epsilon^2})$ and $\epsilon \in [0, \epsilon_1]$.

(b) the differential equation

$$\frac{dz}{d\tau} - \bar{f}_x(\hat{x}_0(\tau,\sigma)) \cdot z = 0 \tag{4.1}$$

has a fundamental matrix $\Phi(\tau, \sigma)$, where σ is treated as a parameter, satisfying the inequality

$$\|\Phi(\tau_2, \sigma)\Phi(\tau_1, \sigma)^{-1}\| \leq a(\sigma)e^{-b(\sigma)(\tau_2-\tau_1)} \tag{4.2}$$

for some positive functions $a(\sigma)$ and $b(\sigma)$ and for $\tau_2 \geq \tau_1 \geq 0$.

Because $\hat{x}_1(\tau, \sigma)$ satisfies (2.22), we have

$$\begin{aligned} \hat{x}_1(\tau, \sigma) &= \Phi(\tau, \sigma) \int_0^\tau \Phi(\eta, \sigma)^{-1} \left[\rho(\hat{x}_0(\eta, \sigma)) - \bar{f}_{1x}(\hat{x}_0(\eta, \sigma)) \cdot \bar{f}(\hat{x}_0(\eta, \sigma))\right] d\eta \\ &\quad - \Phi(\tau, \sigma) \int_0^\tau \Phi(\eta, \sigma) \frac{\partial \hat{x}_0(\eta, \sigma)}{\partial \sigma} \, d\eta. \end{aligned} \tag{4.3}$$

Since $\hat{x}_0(\epsilon t, \epsilon^2 t)$ will be contained in the compact ball K for t in an interval of order $O(\frac{1}{\epsilon^2})$ and for $\epsilon \in [0, \epsilon_1]$, we can find a bound for $\hat{x}_1(\tau, \sigma)$.

$$\begin{aligned} \|\hat{x}_1(\tau, \sigma)\| &\leq \int_0^\tau a(\sigma)e^{-b(\sigma)(\tau-\eta)}[\|\rho(\hat{x}_0(\eta, \sigma)) + \|\bar{f}_{1x}(\hat{x}_0(\eta, \sigma)) \cdot \bar{f}(\hat{x}_0(\eta, \sigma))\|] \, d\eta \\ &\quad + \int_0^\tau a(\sigma)e^{-b(\sigma)(\tau-\eta)} \left\| \frac{\partial \hat{x}_0(\eta, \sigma)}{\partial \sigma} \right\| d\eta. \end{aligned} \tag{4.4}$$

Let

$$M_3 = Max \{\|\rho(z)\| : z \in K\}$$

and

$$M_4 = Max \left\{\|\bar{f}_{1x}(z)\| \cdot \|\bar{f}(z)\| : z \in K\right\}.$$

Then

$$\begin{aligned} \|\hat{x}_1(\tau, \sigma)\| &\leq \int_0^\tau a(\sigma)e^{-b(\sigma)(\tau-\eta)}(M_3 + M_4) \, d\eta \\ &\quad + \int_0^\tau a(\sigma)e^{-b(\sigma)(\tau-\eta)} \left\| \frac{\partial \hat{x}_0(\eta, \sigma)}{\partial \sigma} \right\| d\eta \\ &\leq \frac{a(\sigma)[1 - e^{-b(\sigma)\tau}]}{b(\sigma)}(M_3 + M_4) \\ &\quad + \int_0^\tau a(\sigma)e^{-b(\sigma)(\tau-\eta)} \left\| \frac{\partial \hat{x}_0(\eta, \sigma)}{\partial \sigma} \right\| d\eta. \end{aligned} \tag{4.5}$$

The first term of the last equation is always bounded for $\tau \geq 0$ and $\epsilon \in [0, \epsilon_1]$. Therefore the last equation really does not give us any condition on choosing $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$. Hence the only requirement on $\hat{x}_0(\tau,\sigma)$ is that it is differentiable with respect to σ. Therefore there are many choices for $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma}$. Among them the easiest is that $\frac{\partial \hat{x}_0(\tau,\sigma)}{\partial \sigma} = 0$. This says that $\hat{x}_0(\tau,\sigma) = \hat{x}_0(\tau)$. Now the equation (4.1) becomes

$$\frac{dz}{d\tau} - \bar{f}_x(\hat{x}_0(\tau)) \cdot z = 0 \tag{4.6}$$

which does not depend on σ at all. Therefore the positive functions $a(\sigma)$ and $b(\sigma)$ are replaced by two positive constants a and b. Hence (4.2) becomes

$$\|\Phi(\tau_2) \cdot \Phi(\tau_1)^{-1}\| \leq ae^{-b(\tau_2-\tau_1)} \tag{4.7}$$

for $\tau_2 \geq \tau_1 \geq 0$. Finally we have the following result.

Theorem 3 Suppose $\hat{x}_0(\tau)$ is the solution of the equation

$$\frac{d\hat{x}_0(\tau)}{d\tau} = \bar{f}(\hat{x}_0(\tau)) \tag{4.8}$$

with the initial condition $\hat{x}_0(\tau) = \alpha$ and where $\tau = \epsilon t$. Suppose that the equation

$$\frac{dz}{d\tau} - \bar{f}_x(\hat{x}_0(\tau)) \cdot z = 0 \tag{4.9}$$

has a fundamental matrix $\Phi(\tau)$ having the property that there are positive constants a and b such that

$$\|\Phi(\tau_2)\Phi(\tau_1)^{-1}\| \leq ae^{-b(\tau_2-\tau_1)} \tag{4.10}$$

for all $\tau_2 \geq \tau_1 \geq 0$. Then there exists an $\epsilon_0 > 0$ such that

$$x_{exact}(t,\epsilon) = \hat{x}_0(\epsilon t) + O(\epsilon) \tag{4.11}$$

for $t \geq 0$ and for $\epsilon \in [0, \epsilon_0]$.

Proof: Let

$$R(t,\epsilon) = x_{exact}(t,\epsilon) - \hat{x}_0(\epsilon t). \tag{4.12}$$

Then

$$\begin{aligned} \frac{dR}{dt} &= \epsilon f(t, \hat{x}_0 + R) - \epsilon \bar{f}(\hat{x}_0) \\ &= \epsilon[\bar{f}(\hat{x}_0 + R) - \bar{f}(\hat{x}_0)] + \epsilon \tilde{f}(t, \hat{x}_0 + R) \\ &= \epsilon[\bar{f}(\hat{x}_0 + R) - \bar{f}(\hat{x}_0)] + \epsilon \tilde{f}(t, \hat{x}_0) \\ &\quad + \epsilon[\tilde{f}(t, \hat{x}_0 + R) - \tilde{f}(t, \hat{x}_0)] \\ &= \epsilon \bar{f}_x(\hat{x}_0) \cdot R + \frac{\epsilon}{2} \bar{f}_{xx}(z_1) \cdot R \cdot R \\ &\quad + \epsilon \tilde{f}(t, \hat{x}_0) + \epsilon \tilde{f}_x(t, z_2) \cdot R \end{aligned} \tag{4.13}$$

where z_1 and z_2 are in the compact ball K which is centered at α with radius $\delta > 0$ such that

$$\|\hat{x}_0(\epsilon t)\| + 1 \leq \delta \tag{4.14}$$

for t in an interval of order $O(\frac{1}{\epsilon^2})$ and $\epsilon \in [0, \epsilon_1]$. Rewrite (4.13) and we will have

$$\frac{dR}{dt} - \epsilon \bar{f}_x(\hat{x}_0) \cdot R = \frac{\epsilon}{2} \bar{f}_{xx}(z_1) \cdot R \cdot R + \epsilon \tilde{f}(t, z_2) \cdot R + \epsilon \tilde{f}(t, \hat{x}_0) \tag{4.15}$$

with $R(0, \epsilon) = 0$.

It is known that $R(0,\epsilon) = 0$. Thus there is an interval I, beginning at $t = 0$ and possibly depending on ϵ, over which $\|R(t,\epsilon)\| \leq 1$ for $\epsilon \in [0, \epsilon_1]$ and for $t \in I$. Hence $\hat{x}_0(\epsilon t) + R(t,\epsilon)$ is in K for $t \in I$ and $\epsilon \in [0, \epsilon_1]$. For $\hat{x}_0 + R$ in K, (4.15) is the same differential equation as (4.1) except for the nonhomogeneous part. Therefore as long as $\hat{x}_0 + R$ is in K, we can express $R(t, \epsilon)$ in the following way

$$R(t,\epsilon) \;=\; \epsilon \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \left\{ \frac{1}{2}\bar{f}_{xx}(z_1)\cdot R(s,\epsilon)\cdot R(s,\epsilon) + \tilde{f}_x(s,z_2)\cdot R \right\} ds$$
$$+\, \epsilon \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}(s,\hat{x}_0(\epsilon s))\, ds. \tag{4.16}$$

Let

$$M_5 = Max\left\{ \frac{1}{2}\|\bar{f}_{xx}(z)\| : z \in K \right\}$$

and

$$M_6 = Max\left\{ \|\tilde{f}_x(s,z)\| : s \in R,\ z \in K \right\}.$$

Then

$$\|R(t,\epsilon)\| \;\le\; \epsilon \int_0^t a e^{-b(\epsilon t - \epsilon s)} \left[M_5 \|R(s,\epsilon)\|^2 + M_6 \|R(s,\epsilon)\| \right] ds$$
$$+\, \epsilon \left\| \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}(s,\hat{x}_0(\epsilon s))\, ds \right\|. \tag{4.17}$$

We will find a bound for the second term of the last equation. Since $\tilde{f}$ is mean free, we can write $\tilde{f}$ as its Fourier series

$$\tilde{f}(s,z) = \sum_{n \neq 0} \mathrm{a_n}(z) e^{ins}.$$

Then we can apply the argument from the proof of Lemma 1 to the function $\Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}(s,\hat{x}_0(\epsilon s))$ and obtain

$$
\begin{aligned}
& \epsilon \| \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \sum_{n\neq 0} \mathrm{a}_n(\hat{x}_0(\epsilon s)) e^{ins}\, ds \| \\
= \; & \epsilon \| \sum_{n\neq 0} \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \mathrm{a}_n(\hat{x}_0(\epsilon s)) e^{ins}\, ds \| \\
\leq \; & \epsilon \sum_{n\neq 0} \frac{1}{n} [\|\Phi(\epsilon t)\Phi(\epsilon t)^{-1}\mathrm{a}_n(\hat{x}_0(\epsilon t))\| + \|\Phi(\epsilon t)\Phi(0)^{-1}\mathrm{a}_n(\hat{x}(0))\|] \\
& + \epsilon \| \sum_{n\neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t) \frac{d[\Phi(\epsilon s)^{-1}\mathrm{a}_n(\hat{x}_0(\epsilon s))]}{ds}\, ds \| \\
= \; & \epsilon \sum_{n\neq 0} \frac{1}{n} [a\|\mathrm{a}_n(\hat{x}_0(\epsilon t))\| + ae^{-b\epsilon t}\|\mathrm{a}_n(\alpha))\|] \\
& + \epsilon \| \sum_{n\neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t) \left[\frac{d\Phi(\epsilon s)^{-1}}{ds} \mathrm{a}_n(\hat{x}_0(\epsilon s)) + \Phi(\epsilon s)^{-1} \frac{d\mathrm{a}_n(\hat{x}_0(\epsilon s))}{ds} \right] ds \| \\
= \; & \epsilon \sum_{n\neq 0} \frac{1}{n} [a\frac{k}{n^2} + ae^{-b\epsilon t}\frac{k}{n^2}] \\
& + \epsilon \| \sum_{n\neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t) \left[\frac{d\Phi(\epsilon s)^{-1}}{ds} \mathrm{a}_n(\hat{x}_0(\epsilon s)) + \Phi(\epsilon s)^{-1} \mathrm{a}_n'(\hat{x}_0(\epsilon s)) \frac{d\hat{x}_0(\epsilon s)}{ds} \right] ds \| \\
= \; & \epsilon \sum_{n\neq 0} [\frac{ak}{n^3} + \frac{ae^{-b\epsilon t}k}{n^3}] \qquad (4.18) \\
& + \epsilon \| \sum_{n\neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t) \left[\frac{d\Phi(\epsilon s)^{-1}}{ds} \mathrm{a}_n(\hat{x}_0(\epsilon s)) + \epsilon\Phi(\epsilon s)^{-1} \mathrm{a}_n'(\hat{x}_0(\epsilon s)) \bar{f}_x(\hat{x}_0(\epsilon s)) \right] ds \|.
\end{aligned}
$$

We know that

$$\frac{d\Phi(\epsilon s)}{ds} = \epsilon \bar{f}_x(\hat{x}_0(\epsilon s))\Phi(\epsilon s)$$

and

$$\Phi(\epsilon s)\Phi(\epsilon s)^{-1} = I$$

where I is an $n \times n$ identity matrix. Differentiate the last equation with respect to s and obtain

$$\frac{d\Phi(\epsilon s)\Phi(\epsilon s)^{-1}}{ds} = \frac{dI}{ds} = 0.$$

Hence

$$\frac{d\Phi(\epsilon s)}{ds}\Phi(\epsilon s)^{-1} + \Phi(\epsilon s)\frac{d\Phi(\epsilon s)^{-1}}{ds} = 0.$$

Thus

$$\epsilon \bar{f}_x(\hat{x}_0(\epsilon s))\Phi(\epsilon s)\Phi(\epsilon s)^{-1} = -\Phi(\epsilon s)\frac{d\Phi(\epsilon s)^{-1}}{ds}.$$

Finally we have

$$\frac{d\Phi(\epsilon s)^{-1}}{ds} = -\epsilon\Phi(\epsilon s)^{-1}\bar{f}_x(\hat{x}_0(\epsilon s)). \tag{4.19}$$

Hence

$$\begin{aligned}
&\epsilon\|\int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}(s,\hat{x}_0(\epsilon s))\,ds\| \\
\leq\ & \epsilon\sum_{n\neq 0}[\frac{ak}{n^3} + \frac{ake^{-b\epsilon t}}{n^3}] \\
&+ \epsilon^2\|\sum_{n\neq 0}\int_0^t \frac{e^{ins}}{in}\Phi(\epsilon t)\Phi(\epsilon s)^{-1}[\mathrm{a}_n'(\hat{x}_0(\epsilon s))\bar{f}_x(\hat{x}_0(\epsilon s)) - \bar{f}_x(\hat{x}_0(\epsilon s))\mathrm{a}_n(\hat{x}_0(\epsilon s))]\,ds\| \\
=\ & \epsilon\sum[\frac{ak}{n^3} + \frac{ake^{-b\epsilon t}}{n^3}] \qquad (4.20)\\
&+ \epsilon^2\sum_{n\neq 0}\int_0^t \frac{ae^{-b\epsilon(t-s)}}{n}[\|\bar{f}_x(\hat{x}_0(\epsilon s))\mathrm{a}_n(\hat{x}_0(\epsilon s))\| + \|\mathrm{a}_n'(\hat{x}_0(\epsilon s))\bar{f}_x(\hat{x}_0(\epsilon s))\|]\,ds
\end{aligned}$$

Let $M_7 = Max\left\{\|\bar{f}_x(z)\| : z \in K\right\}$. Then we have

$$\begin{aligned}
&\epsilon\|\int_0^t \Phi(\epsilon t)\Phi(\epsilon)^{-1}\tilde{f}(s,\hat{x}_0(\epsilon s))\,ds\| \\
\leq\ & \epsilon\sum_{n\neq 0}[\frac{ak}{n^3} + \frac{ake^{-b\epsilon t}}{n^3}] + \epsilon^2\sum_{n\neq 0}\int_0^t \frac{M_7}{n}ae^{-b\epsilon(t-s)}\left[\frac{k}{n^2} + \frac{l}{n}\right]ds \\
=\ & \epsilon\sum_{n\neq 0}[\frac{ak}{n^3} + \frac{ake^{-b\epsilon t}}{n^3}] + \epsilon\sum_{n\neq 0}\frac{aM_7}{b}\left[\frac{k}{n^3} + \frac{l}{n^2}\right][1 - e^{-b\epsilon t}] \\
\leq\ & \epsilon C_1 \qquad (4.21)
\end{aligned}$$

where $C_1 = \sum_{n\neq 0}[\frac{ak}{n^3} + \frac{ak}{n^3} + \frac{aM_7}{b}(\frac{k}{n^3} + \frac{l}{n^2})]$. Therefore as long as $\|R(t,\epsilon)\| \leq 1$ we have

$$\begin{aligned}
\|R(t,\epsilon)\| &\leq \epsilon C_1 + \epsilon\int_0^t ae^{-b\epsilon(t-s)}[M_5\|R(s,\epsilon)\|^2 + M_6\|R(s,\epsilon)\|]\,ds \\
&= \epsilon C_1 + \epsilon\int_0^t ae^{-b\epsilon(t-s)}[M_5 + M_6]\|R(s,\epsilon)\|\,ds \qquad (4.22)
\end{aligned}$$

By Gronwall's inequality we have

$$\begin{aligned} \|R(t,\epsilon)\| &\leq \epsilon C_1 + \epsilon \frac{C_1 a(M_5+M_6)}{b} e^{\frac{a(M_5+M_6)}{b}} \left(1 - e^{-b\epsilon t}\right) \\ &= \epsilon C_2 \end{aligned} \tag{4.23}$$

where $C_2 = C_1 + \frac{C_1 a(M_5+M_6)}{b} e^{\frac{a(M_5+M_6)}{b}}$.

Pick $\epsilon_0 > 0$ such that $\epsilon_0 C_2 < \frac{1}{2}$. Then (4.23) implies that $\|R(t,\epsilon)\| \leq \frac{1}{2}$ for as long as $\|R(t,\epsilon)\| \leq 1$, provided $\epsilon \in [0,\epsilon_0]$. Since R begins at zero if $\|R(t,\epsilon)\| \leq \frac{1}{2}$ ever fails, it must fail before $\|R(t,\epsilon)\| \leq 1$, since $\frac{1}{2}$ is less than 1. And yet this is exactly what cannot happen, since we have shown that as long as $\|R(t,\epsilon)\| \leq 1$, $\|R(t,\epsilon)\| \leq \frac{1}{2}$ cannot fail. Therefore both $\|R(t,\epsilon)\| \leq 1$ and $\|R(t,\epsilon)\| \leq \frac{1}{2}$, and all the equations in between, must hold for all $t \geq 0$, provided $\epsilon \in [0,\epsilon_0]$. That is

$$\|x_{exact}(t,\epsilon) - \hat{x}_0(\epsilon t)\| \leq \epsilon C_2 \tag{4.24}$$

for all $t \geq 0$ and ϵ in $[0,\epsilon_1]$.

Q.E.D.

We already know that under the assumptions (a) and (b) that

$$y = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^k x_k$$

is uniformly ordered for $k \geq 1$. Now we will show that

$$y = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^k x_k \tag{4.25}$$

is a uniformly valid asymptotic approximate solution of order $O(\epsilon^{k+1})$ to $x_{exact}(t,\epsilon)$ for all $t \geq 0$.

Before we show the result, we need to know that when we solve x_i completely we must use the average of the order $O(\epsilon^{i+1})$ term in the remainder of the equation

$$\frac{d(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^i x_i)}{dt} - \epsilon f(t, x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots + \epsilon^i x_i) \tag{4.26}$$

for each $i \geq 1$. Therefore we have

$$\begin{aligned} \frac{dy}{dt} - \epsilon f(t,y) &= \epsilon^{k+1} \tilde{f}_{k+1}(t, x_0, x_1, \cdots, x_k) \\ &\quad + \epsilon^{k+2} f_{k+2}(t, x_0, x_1, \cdots, x_k) \end{aligned} \tag{4.27}$$

where $\tilde{f}_{k+1}$ is a mean free function and f_{k+2} is the remainder of order ϵ^{k+2} and higher order terms.

Let

$$R(t,\epsilon) = x_{exact}(t,\epsilon) - y.$$

Then

$$\begin{aligned} \frac{dR}{dt} &= \epsilon f(t, y+R) - \frac{dy}{dt} \\ &= \epsilon[f(t,y+R) - f(t,y)] + [\epsilon f(t,y) - \frac{dy}{dt}] \\ &= \epsilon[f_x(t,y) \cdot R + \frac{1}{2} f_{xx}(t,w_1) \cdot R \cdot R] \\ &\quad - [\epsilon^{k+1} \tilde{f}_{k+1} + \epsilon^{k+2} f_{k+2}] \\ &= \epsilon[f_x(t, x_0 + \cdot + x_k) \cdot R + \frac{1}{2} f_{xx}(t,w_1) \cdot R \cdot R] \\ &\quad - \epsilon^{k+1} \tilde{f}_{k+1} - \epsilon^{k+2} f_{k+2} \\ &= \epsilon[f_x(t,x_0) + f_{xx}(t,w_2)(\epsilon x_1 + \cdots + \epsilon^k x_k)] \cdot R + \frac{\epsilon}{2} f_{xx}(t,w_1) \cdot R \cdot R \\ &\quad - \epsilon^{k+1} \tilde{f}_{k+1} - \epsilon^{k+2} f_{k+2} \\ &= \epsilon f_x(t,x_0) \cdot R + \epsilon f_{xx}(t,w_2)(y - x_0) \cdot R + \frac{\epsilon}{2} f_{xx}(t,w_1) \cdot R \cdot R \end{aligned}$$

$$
\begin{aligned}
&- \epsilon^{k+1}\tilde{f}_{k+1} - \epsilon^{k+2} f_{k+2} \\
= \;& \epsilon \bar{f}_x(t, x_0) \cdot R \\
&+ \epsilon[\tilde{f}_x(t, x_0) + f_{xx}(t, w_2)(y - x_0)] \cdot R \\
&+ \frac{\epsilon}{2} f_{xx}(t, w_1) \cdot R \cdot R + \; - \epsilon^{k+1}\tilde{f}_{k+1} - \epsilon^{k+2} f_{k+2}.
\end{aligned} \tag{4.28}
$$

If we move the first term on the right hand side to the left hand side, then we can see that $R(t, \epsilon)$ also satisfies (4.1) except for the nonhomogeneous part. Therefore

$$
\begin{aligned}
R(t, \epsilon) \;=\; & \epsilon \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}[\tilde{f}_x(s, x_0(\epsilon s)) + f_{xx}(s, w_2)(y - x_0)] \cdot R\, ds \\
& + \frac{\epsilon}{2} \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} f_{xx}(s, w_1) \cdot R \cdot R\, ds \\
& - \epsilon^{k+1} \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \tilde{f}_{k+1}\, ds \\
& - \epsilon^{k+2} \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} f_{k+2}\, ds
\end{aligned} \tag{4.29}
$$

We now can apply Lemma 1 to $\epsilon^{k+1} \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}_{k+1}\, ds$. Because $\tilde{f}_{k+1}$ is mean free, it has a Fourier series

$$
\tilde{f}_{k+1}(t, y) = \sum_{n \neq 0} \mathrm{b_n}(y) e^{int},
$$

where $\mathrm{b_n}$ is a function of $y = x_0 + \epsilon x_1 + \cdots + \epsilon^k x_k$ for each $n \neq 0$. Thus we have the inequality

$$
\begin{aligned}
& \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1}\tilde{f}_{k+1}(s, y)\, ds \\
= \;& \sum_{n \neq 0} \frac{\Phi(\epsilon t)\Phi(\epsilon t)^{-1}\mathrm{b_n}(y) - \Phi(\epsilon t)\Phi(0)^{-1}\mathrm{b_n}(\alpha)}{n} \\
& - \sum_{n \neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t) \frac{d(\Phi(\epsilon s)^{-1}\mathrm{b_n}(y))}{ds}\, ds \\
= \;& \sum_{n \neq 0} \frac{a\mathrm{b_n}(y) - a e^{-b\epsilon t}\mathrm{b_n}(\alpha)}{n}
\end{aligned}
$$

$$- \epsilon \sum_{n\neq 0} \int_0^t \frac{e^{ins}}{in} \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \left\{ -\bar{f}_x(x_0)\mathrm{b}_n(y) + \mathrm{b}_n(y)\bar{f}_x(x_0) \right\} ds. \quad (4.30)$$

Let

$$M_8 = Max \left\{ \|f_{k+2}(t, z_1, \cdots, z_k)\| : t \in [0, 2\pi], z_i \in K \ for \ i = 1, 2, \cdots, k \right\}$$

and

$$M_9 = Max \left\{ \|\frac{1}{2} f_{xx}(t, z)\| : t \in [0, 2\pi], z \in K \right\}.$$

Because f_{xx} and f_{k+2} are 2π-periotic, we can find these bounds. Therefore we obtain

$$\begin{aligned}
& \| \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \tilde{f}_{k+1}(s, y)\, ds \| \\
\leq \; & \sum_{n\neq 0} \left(\frac{ak}{n^3} + \frac{al}{n^2} \right) \\
& + \epsilon \sum_{n\neq 0} \int_0^t \frac{ae^{-b\epsilon t}}{n} M_8 \left(\frac{k}{n^2} + \frac{l}{n} \right) ds \\
\leq \; & \sum_{n\neq 0} \left(\frac{ak}{n^3} + \frac{al}{n^2} \right) \\
& + \frac{aM_7}{b} \left(1 - e^{-b\epsilon t} \right) \sum_{n\neq 0} \left(\frac{k}{n^3} + \frac{l}{n^2} \right) \\
\leq \; & \sum_{n\neq 0} \left[\frac{ak}{n^3} + \frac{al}{n^2} \right] + \frac{aM_8}{b} \sum_{n\neq 0} \left[\frac{k}{n^3} + \frac{l}{n^2} \right] \\
= \; & C_3 \qquad (4.31)
\end{aligned}$$

where

$$C_3 = \sum_{n\neq 0} [\frac{ak}{n^3} + \frac{al}{n^2}] + \frac{aM_8}{b} \sum_{n\neq 0} [\frac{k}{n^3} + \frac{l}{n^2}].$$

Then we have a bound for the ϵ^{k+1} term and which is

$$\begin{aligned}
& \epsilon^{k+1} \| \int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} \tilde{f}_{k+1}(s, y)\, ds \| \\
\leq \; & \epsilon^{k+1} C_3.
\end{aligned}$$

Also we have

$$\begin{aligned}
\epsilon^{k+2}\|\int_0^t \Phi(\epsilon t)\Phi(\epsilon s)^{-1} f_{k+2}(s,y)\,ds\| &\leq \epsilon^{k+2}\int_0^t a e^{-b\epsilon(t-s)} M_8\,ds \\
&\leq \epsilon^{k+1}\frac{aM_8}{b}\left(1-e^{-b\epsilon t}\right) \\
&\leq \epsilon^{k+1} C_4 \qquad (4.32)
\end{aligned}$$

where $C_4 = \frac{aM_8}{b}$ and for all $t \geq 0$. Therefore

$$\begin{aligned}
\|R(t,\epsilon)\| \leq\ & (C_3\epsilon^{k+1} + C_4\epsilon^{k+1}) \\
& + \epsilon\int_0^t a e^{-b\epsilon(t-s)} M_9 \|R(s,\epsilon)\|^2\,ds \qquad (4.33) \\
& + \epsilon\int_0^t a e^{-b\epsilon(t-s)} [\|\tilde{f}_x(s,x_0)\| + \|f_{xx}(s,w_2)(y-x_0)\|]\cdot\|R(s,\epsilon)\|\,ds.
\end{aligned}$$

Let

$$M_{10} = Max\left\{\|\tilde{f}_x(t,z)\| + \|f_{xx}(t,z)(y-x_0)\| : t\in[0,2\pi], z\in K\right\}.$$

Then as long as $\|R(t,\epsilon)\| \leq 1$, we have

$$\begin{aligned}
\|R(t,\epsilon)\| \leq\ & \epsilon^{k+1}[C_3+C_4] \\
& + \epsilon\int_0^t a e^{-b\epsilon(t-s)}[M_{10}+M_9]\|R(s,\epsilon)\|\,ds. \qquad (4.34)
\end{aligned}$$

By Gronwall's inequality we obtain

$$\begin{aligned}
\|R(t,\epsilon)\| &\leq \epsilon^{k+1}[C_3+C_4] \\
&\quad + \epsilon^{k+1}\frac{[C_3+C_4]}{b} a[M_{10}+M_9] e^{\frac{a[M_{10}+M_9]}{b}}\left(1-e^{-b\epsilon t}\right) \\
&\leq \epsilon^{k+1} C_5, \qquad (4.35)
\end{aligned}$$

where $C_5 = C_3 + C_4 + \frac{[C_3+C_4]}{b} a[M_{10}+M_9] e^{\frac{a[M_{10}+M_9]}{b}}$, for all $t \geq 0$. Then use the bootstrapping again, we have

$$\|R(t,\epsilon)\| \leq \epsilon^{k+1} C_5$$

for all $t \geq 0$ and $\epsilon \in [0, \epsilon_0]$ for some $\epsilon_0 > 0$. That is to say that

$$\|x_{exact}(t, \epsilon) - y\| = O(\epsilon^{k+1}) \tag{4.36}$$

for all $t \geq 0$ and for $\epsilon \in [0, \epsilon_1]$.

Q.E.D.

CHAPTER 5. CONCLUSION

In this thesis, we have studied the 3-scale method which is applied to a periodic first order ordinary differential equation. We expect that the 3-scale method yields a uniformly valid approximate solution on an interval of order $O(\frac{1}{\epsilon^2})$. In chapter 2, we give an example which shows that the 3-scale method fails to give an approximate solution on an interval of order $O(\frac{1}{\epsilon^2})$.

In chapter 1, we give the weaker definitions of "uniform ordering" and "uniform validity". This will allow us to apply the method of multiple scales to more applications.

In chapter 3, we have shown that if $\bar{f}(z) = 0$, then the 3-scale method and the method of averaging produce the asymptotic equivalent approximations on an interval of order $O(\frac{1}{\epsilon^2})$.

In chapter 4, we impose two assumptions and get the corresponding result as the Sanchez-Palencia Theorem. But we must require that $\hat{x}_0(\epsilon t)$ is bounded for all $t \geq 0$. If $\hat{x}_0(\epsilon t)$ is bounded for $t \in I_\epsilon$ with $I_\epsilon \neq [0, \infty)$, then the Theorem 3 is true only for $t \in I_\epsilon$ and not true for all $t \geq 0$.

The method of multiple scales and the method of averaging come from the different ideas. People are kind of expecting that they produce asymptotic equivalent approximate solutions to the same problem. We know that the 2-scale method and the method of averaging yield the same zeroth order approximation. Not many known results tell us the relations between the 3-scale method and the method of averaging. We hope that this thesis will move one step in this direction.

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